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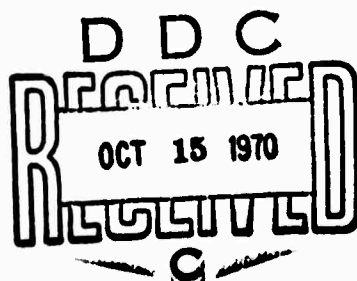# WHAT IF UTILITY FUNCTIONS DO NOT EXIST?

Fred S. Roberts

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Discusses alternatives to the strict decisionmaking goal of ranking all alternatives or, equivalently, obtaining a utility function. For many military problems, the best information available is the combined judgments of experts. Often, however, the preferences of decisionmakers are too inconsistent or ambiguous to permit a complete ranking of alternatives or a utility function. Optional methods for dealing with such a situation include (1) finding a procedure through which preferences can be modified to obtain a utility function; (2) using the utility assignment that best approximates a utility function; or (3) modifying the demands on utility functions. This study emphasizes the third alternative, and describes it in terms of techniques from the theory of measurement, recently developed by behavioral scientists, that facilitate decisionmaking where no utility functions exist.	Mathematics Social Sciences Decisionmaking Military Planning

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PREFACE

For many military problems, the best information available is the judgments of knowledgeable individuals, and these judgments must somehow be combined to produce rational decisions.

Often the preferences of decisionmakers are inconsistent or not sufficiently well-defined to obtain a complete ranking of alternatives, or equivalently a utility function. Such a situation often arises if there is more than one decisionmaker and preferences are obtained by lumping, or if the alternatives are considered from several different points of view or under several different contingencies. The situation arises in such varied military problems as choosing officers for promotion (M candidates are evaluated by each of N different judges) and choosing alternative transport systems (M alternative systems are evaluated under N different situations (contingencies) in which they might be used).

This report discusses alternatives to the strict decisionmaking goal of obtaining a full-fledged ranking of alternatives or a utility function. It describes techniques from the theory of measurement, developed in recent years

by behavioral scientists, which help in decisionmaking situations where no utility functions exist.

Related material may be found in RM-5957-PR, RM-6115-PR, RM-6118-PR, and RM-6299-ARPA.

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SUMMARY

The ultimate aim of the decisionmaking art can be stated as follows: Given the preferences of your decision-makers, obtain a complete ranking of the alternatives which reflects these preferences; equivalently, assign numbers to the various alternatives so that alternative x is preferred to alternative y precisely when x gets a higher number than y. The number assigned to an alternative is usually called its utility, and sometimes its worth, and the assignment is a utility function.

Often the preferences of decisionmakers are sufficiently inconsistent so as to preclude the existence of such a utility function. This report discusses several alternatives for dealing with the situation where no utility function exists, in particular

1. Finding a procedure whereby we can modify or re-define or make explicit our preferences in the course of decisionmaking in order to obtain a utility function.

2. Settling for a utility assignment which best approximates a utility function.

3.   Modifying the demands on utility functions.

This study emphasizes the third alternative, and describes it in some detail in the context of the theory of measurement developed in recent years by behavioral scientists. The main theme of the measurement theory approach is that to measure preferences, we assign numbers (not necessarily utilities) which reflect these preferences in some precisely defined way. Once having assigned numbers, we can use the full power of our mathematical techniques for dealing with and manipulating numbers in order to understand the preferences. In this spirit, preferences may also be "measured" by assigning other concrete mathematical objects to alternatives, including vectors, intervals on the line, and random variables.

The measurement theory approach is illustrated by developing in some detail the concept of dimension of a partial order. Even if we cannot assign numerical utilities or worth values which reflect preferences in the classical sense, from the measurement theory point of view we can still learn a lot about the preferences by finding several measures of worth so that a given alternative x is preferred to an alternative y if and only if x is ranked higher than y on each of the worth scales. If such measures can be found,

it follows that the preferences define a partial order, and the smallest number of such scales needed is called the dimension of the partial order. The ultimate aim of the decisionmaking art can now be restated as that of transforming partially ordered preferences into preferences which have as low a dimension as possible. If one-dimensional preferences (those amenable to classical utility functions) cannot be found, then the next best thing is to search for two-dimensional preferences. Several conditions under which a partial order is two-dimensional are described.

ACKNOWLEDGEMENTS

CONTENTS

WHAT IF UTILITY FUNCTIONS DO NOT EXIST?

1. UTILITY FUNCTIONS OFTEN DO NOT EXIST

The ultimate aim of the decisionmaking art seems to be: Given the preferences of your decisionmakers, obtain a complete ranking of the alternatives which reflects these preferences. Equivalently, this aim may be stated as follows: Assign numbers to the various alternatives so that alternative x is preferred to alternative y _precisely_ when x gets a higher number than y.[*] The number assigned to an alternative is usually called its _utility_, and sometimes its _worth_, and the assignment is a _utility function_.

Formally, if X is the set of alternatives and < is a binary relation on X representing (observed) preference, then we require a function u:X ⟶ Reals so that for all x, y ∈ X,

(1) $\qquad x < y \longleftrightarrow u(x) < u(y).$

We shall say that x is _weakly preferred_ to y if not x < y, and we are _indifferent_ between x and y if we prefer neither.

Often the preferences (or evaluations) of decisionmakers are sufficiently inconsistent (some might say "irrational") so as to preclude the existence of such a

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[*] These two aims are equivalent so long as the set of alternatives is finite, as it is in all practical examples.

worth or utility assignment. We mention just one argument given in support of this empirical fact. If we can assign numbers satisfying (1), then it certainly follows that our weak preferences (and also our indifferences) must be transitive: If x is weakly preferred to y and y to z, then x must be weakly preferred to z. The transitivity of weak preference (and of indifference) has been attacked in the literature, with such attacks going back to the economist Armstrong [1, 2, 3, 4]. A whole series of models for preference with nontransitive indifference is described in Roberts [34] and in Fishburn [13].

This paper deals with the question: What is one to say and do in the disturbing situation where no utility function exists? Before turning to this question, we should note that much of utility theory deals with preferences between complex alternatives. Thus, combinations or mixtures of objects in X are allowed, and we require of the utility function that it preserve combinations.

For example, let x and y be certain alternatives, let p be a number between 0 and 1, and let xpy denote the uncertain alternative "x with probability p and y with probability 1-p." If you are indifferent between alternatives x and y, then for every alternative z, you "ought" to be indifferent between xpz and ypz. More generally, your utility function ought to satisfy

$$u(xpy) = p \cdot u(x) + (1-p) \cdot u(y),$$

for all x y ε X and p between 0 and 1. Studies of utility functions satisfying this condition, often called <u>cardinal util-</u><u>ity functions</u>, originated with von Neumann and Morgenstern [40].

In our discussion we shall not deal with combinations or mixtures and the like. We shall concentrate simply on utility functions preserving the relation < in the sense of equation (1), the so-called <u>ordinal utility functions</u>.

If no ordinal utility function exists, one approach to the decisionmaking problem is to describe a procedure whereby we can modify or redefine or make explicit our preferences in the course of decisionmaking in order to become more "rational" (i.e., so that such a utility function will exist). A second approach is to settle for a utility assignment which best approximates equation (1).

As for the first approach, that of modifying preferences to obtain rationality, this is at least the spirit of many papers which suggest the following method. Build a utility function (or some sort of function assigning numbers to alternatives, to be more accurate) from some evidence of our preferences, and then use the resulting function to <u>define</u> preference. This is a "normative" (as opposed to "descriptive") approach. It emphasizes how we should choose, what our preferences should be. This approach (and the normative-descriptive distinction[*]) is emphasized, for example, in a long recent Rand study by Raiffa [31]. A

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[*]For more discussion of this distinction, see Marschak [27] and the survey article [15] by Fishburn.

second paper illustrating this approach is one by Miller
[29]. Miller's basic idea is to reduce a complex problem
to several simpler ones. Thus, he suggests breaking each
alternative into several simple facets or components. For
example, alternative transport systems can be studied
under different contingencies. And if a new Ph.D. is
selecting a job, he might consider various factors in
assessing a particular job offer, such as monetary compen-
sation, geographic location, travel requirements, and
"nature of work." Each alternative is measured or scaled
(for its performance) on each facet. The utility or worth
of an alternative on a given facet is obtained by using
the numerical measure or scale value on the facet. And,
finally, the utility of an alternative is obtained by
adding (in a weighted sum) the utilities on the different
facets.

This utility function is then used to define prefer-
ence, and so there can be no problem about equation (1)
being satisfied. On the other hand, if preferences among
alternatives are gathered first, and then this procedure
purports to produce a utility function reflecting them,
there is considerable difficulty. If no assignment satis-
fying (1) exists, then obviously neither this nor any other
procedure is going to find one. And even if an assignment
satisfying (1) exists, this procedure might not produce
such an assignment.

In most of the literature, there is considerable stress on outlining procedures for obtaining utilities, and not enough stress on specification of conditions under which such procedures will "work." Yet, it is costly to invest time, energy, and resources on a decisionmaking procedure without knowing it is (at least reasonably likely) to lead to success.

A more difficult method of modifying preferences to obtain rationality than that described above is first to modify preferences and then to build a utility function reflecting them. Perhaps the only significant work on modifying preferences in the course of decisionmaking without first finding utility functions deals with a group decisionmaking framework. Here, such techniques as Delphi have been studied (see [9]), with the ultimate aim of producing better decisions in groups by having some controlled interaction. The basic idea is that by allowing controlled interactions, we stand a chance of making our preferences more rational.

A second approach, if no utility function exists, is to try to seek utility assignments which best approximate equation (1). There does not seem to have been much work done on this approach. One method begins by obtaining complete rankings on several different facets or under several different contingencies (as in the Miller [29] approach above). Then the overall ranking is taken to be that ranking which

is the "best possible" consensus. Methods for defining
the consensus ranking are given in Kendall [22], Friedman
[17], and in Kemeny and Snell [21]. The best possible
consensus will give rise to a utility function which hope-
fully approximates equation (1) "well enough."

In our (almost obsessive) search for a utility function
satisfying equation (1), we have perhaps missed one approach
to the dilemma that our everyday preferences are not always
amenable to such a representation. Entirely different from
the two described above, this third approach is simply to
modify the demands on the utility function. The outline
of this approach is the subject of the next section.

2. MODIFYING DEMANDS ON UTILITY: THE "MEASUREMENT THEORY" APPROACH

This third approach is perhaps best understood in terms of the "theory of measurement." The point of view we shall take about measurement is based on the ideas of Scott and Suppes [37] and Suppes and Zinnes [38]. The latter is an extensive, elementary introduction to the theory.

We all agree that measurement has something to do with the assignment of numbers. Analysis of such paradigm cases as the measurement of mass and the measurement of temperature indicates more, namely that measurement is the assignment of numbers which preserves certain observed relations. Thus, measurement of mass is the assignment of numbers which preserves the relation "heavier than" and measurement of temperature is the assignment of numbers which preserves the relation "warmer than." In this sense, utility is the measurement of preference, i.e., the assignment of numbers which preserves the relation "preferred to."

Scott and Suppes [37] formalize measurement as follows. We start with an observed (binary, say) relation R on a set of objects or alternatives X. Then, choosing an appropriate relation π on the real numbers, e.g., <, we try to set up a mapping u from X into the reals which π-preserves the relation R, i.e., so that for all x, y $\in$ X,

(2) $xRy \longleftrightarrow u(x) \ \pi \ u(y)$.

The point of assigning numbers is that we know a great deal about their properties and so we can understand the observed relation much better if we can replace it by a concrete relation on numbers. But this purpose will have been accomplished if we replace the observed relation by any useful concrete relation on the real numbers, and it does not have to be <.

To give a helpful example here, let us return to the point we made earlier, namely that our weak preferences (or indifferences) are not necessarily transitive. In particular, indifference is not transitive. A very good example of Luce [25] in support of this point is the following. We certainly have a (strong) preference between a cup of coffee with one spoon of sugar and a cup with four spoons. But if we add sugar to the first cup at the rate of 1/100 of a gram, we will undoubtedly be indifferent between successive cups. Transitivity of indifference would imply indifference between the cups with one spoon and four spoons as well.

This sort of example indicates that indifference might correspond more to "closeness," a nontransitive relation, than to equality, as the utility representation of the form (1) implies. Maybe we should modify our demands in measurement of preference to at least take this idea into account, and thus demand an assignment of numbers so that y is preferred to x if and only if the number assigned to y is

not only larger than the number assigned to x, but "suf-
ficiently larger" so that we can tell them apart. Measuring
"sufficiently larger" by some positive number δ, Scott and
Suppes [37] formalize this idea by demanding a real-valued
function u on the set of alternatives so that for all
alternatives x, y,

(3) $x < y \longleftrightarrow u(x) < u(y) + \delta$.

The point is that frequently a utility assignment satisfying
(3) can be found when an assignment satisfying (1) cannot.
This still gives us a working tool in making decisions,
with the option that if u(x) and u(y) are "close," we can-
not decide. It is certainly fairer to describe preference
data this way than in a way which imposes a preference when
there really is none.

Scott and Suppes give necessary and sufficient condi-
tions, in terms of the relation <, for obtaining a represen-
tation (3). A relation so representable is called a semi-
order, a notion going back to Luce [25]. They also give a
direct procedure for obtaining the function u if it exists.
These are two of the major concerns of measurement theorists:
to exhibit conditions under which useful representations such
as (3) are attainable, and to provide constructive procedures
for attaining them. Further references on semiorders are Suppes

and Zinnes [38], Roberts [32-36], Holland [19, 20], Domotor [10], Krantz [23], and Fishburn [12, 14].

Of course, our preferences will often not be representable in the form (3). But once we take the broad attitude of measurement theory, then we may try other representations; our tools in decisionmaking are limited only by our imagination. Simply find useful concrete relations π on the real numbers which reflect observed preference in the sense of equation (2), i.e., so that

$$x < y \longleftrightarrow u(x) \ \pi \ u(y).$$

(In the semiorder case, $a \ \pi \ b \longleftrightarrow a < b + \delta$.) For that matter, why limit ourselves to numbers $u(x)$? Why not assign other concrete mathematical objects $u(x)$ to alternatives x so that concrete relations on the $u(x)$'s reflect observed relations (preference relations) on the x's. In particular, the $u(x)$ may be vectors rather than numbers. This is measurement with several numerical assignments. We shall give an example of this kind of measurement in Sec. 3. A more radical idea is the following. For each x, find an interval on the real line $u(x)$, to be interpreted as the range of values of x or something similar, so that for all $x, y \in X,$

$$x < y \longleftrightarrow u(x) \ \circleddash \ u(y),$$

where u(x) $\bigotimes$ u(y) means a < b for all a ϵ u(x), b ϵ u(y).
This assignment has been studied in Fishburn [12, 14] and
in Roberts [34]; and this more general approach to measure-
ment theory is advocated by Krantz [24].

To generalize even further at this point, one can
learn a great deal about preferences by taking as basic
data something other than the relation <. To give an
example, the theory of probabilistic consistency takes as
basic data a collection of numbers p(x, y) representing the
frequency with which x is preferred to y. The theory, recog-
nizing that absolute consistency or absolute rationality is
hardly ever attained, tries to define what it means for the
decisionmaker(s) to be probabilistically consistent. A
typical condition of probabilistic consistency is the
strong utility model, which is satisfied if there are random
variables u_x so that for all x, y ϵ X,

$$p(x, y) = \text{Prob} [u_x > u_y].$$

This is a generalization of (1), and the problem is to find
necessary and sufficient conditions on the data p(x, y)
for the existenoe of such random variables. The theory of
probabilistic consistency is developed in Block and Marshak
[8], Luce and Suppes [26], and Roberts [32].

In summary, the job of the measurement theorist is
this: find useful representations, specify conditions

sufficient and/or necessary[*] for these representations, and
specify constructive procedures for obtaining the represen-
tations.[**] The attitude of measurement theory toward measure-
ment of preference and toward decisionmaking is the following.
Accept the fact that our preferences may not be rational
or consistent in the usual utility-function sense. Much is
to be gained by translating these preferences into concrete
relations on numbers or other known mathematical objects,
because then we get an accurate and understandable picture
of our preferences, and also can use the full range of our
mathematical knowledge. Build up a repertoire of useful
representations to try out. Then, rather than try to twist
preferences into a given mold, report them in an insightful
and useful way, and make the best of the information origin-
ally given you. We illustrate this measurement theory
approach in some detail in the next section.

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[*]The sufficient conditions are useful in practice to
tell us when a given mathematical model (given representation
like (2)) can be assumed. The necessary conditions are
useful in practice to tell us when a given model should not
be assumed, namely if it implies a condition which is clearly
unsatisfactory.

[**]It is also important to discuss uniqueness of the
representations, for the uniqueness theorems define the
type of scale involved. (Cf. Suppes and Zinnes [38] for an
elaboration of the relation between uniqueness of the represen-
tation and scale type.)

3. TWO-DIMENSIONAL PARTIAL ORDERS: A SPECIFIC EXAMPLE

To give a detailed example illustrating the points made in the previous section, we shall use the measurement theory approach to develop the concept of two-dimensional partial orders, and discuss the relevance of this relatively unknown concept for decisionmaking.

If the preferences are asymmetric ($x < y$ implies $\sim y < x$) and transitive ($x < y$ and $y < z$ imply $x < z$), then they are said to constitute a partial ordering of the alternatives.

Let us suppose that we cannot assign numerical worth values which reflect the preferences in the sense of (1). From the measurement theory point of view, we can still learn a lot about the preferences by finding several measures of worth so that a given alternative x is preferred to an alternative y if and only if x is ranked higher than y on each of the worth scales. (The different scales may represent different attributes or aspects or contingencies, etc., as in the Miller [29] method discussed above.) If such measures can be found, it follows that the preferences are transitive and therefore define a partial order. The smallest number of such scales needed is called the dimension* of the partial order. The ultimate aim of the decisionmaking art can be restated as that of transforming our partially ordered preferences into one-dimensional

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*This definition differs just slightly from that formally adopted below.

preferences. Short of this, it is useful to minimize the dimension of the preferences, and in particular to search for the next best thing, two-dimensional partial orders. The main purpose of this section is to analyze conditions under which a partial order is two-dimensional. Since conditions characterizing the one-dimensional partial orders are known (see below), it is sufficient to characterize the class of partial orders with dimension at most two.

In the following, we search for conditions on $\prec$ which are necessary and sufficient for the existence of two real-valued functions f and g on X so that for all x, y ϵ X,

(4) $x \prec y$ if and only if $[f(x) < f(y)$ and $g(x) < g(y)]$.

We should remark here that (4) can be thought of as a natural generalization of (1). For, let us define $<$ on R^{2*} by

$(a_1, b_1) < (a_2, b_2)$ if and only if $[a_1 < a_2$ and $b_1 < b_2]$.

Then the existence of functions f and g satisfying (4) is

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*R^2 is the real plane.

equivalent to the existence of a function $u: X \rightarrow R^2$ so that for all $x, y \in X$,

$$x < y \quad \text{if and only if} \quad u(x) < u(y).$$

Necessary and sufficient conditions on $(X, <)$ for the representation (1) have long been known. If X is countable, then $<$ satisfies equation (1) if and only if it is a so-called weak order. (Proofs of this and its extension to X of arbitrary cardinality are given, for example, by Milgram [28], Birkhoff [7], Luce and Suppes [26], and Fishburn [16].)

To return to the representation (4), we begin with some definitions. With a few obvious exceptions, binary relations are here defined on a given set X. We recall that a binary relation $<$ is a <u>partial order</u> if it is asymmetric and transitive. $<$ is a <u>linear order</u> if it is a partial order which is complete ($x \neq y$ implies $x < y$ or $y < x$).

Finally $x \sim y$ means that $x \not< y$ and $y \not< x$. $\sim$ is variously referred to as indifference, matching, similarity, and so forth. We call it <u>indifference</u>. $x \sim y$ means that $x \neq y$ and $x \sim y$. This relation is called <u>incomparability</u>.

Intersections of binary relations are defined as usual:

$(x, y) \in \bigcap_A <_a$ if and only if $(x, y) \in <_a$ [i.e., $x <_a y$]

for all $a \in A$. Clearly, the intersection of a set of linear

orders is a partial order. Conversely, every partial order

can be realized as the intersection of a set of linear

orders, for by Szpilrajn's extension theorem [39], if

$x \sim y$, there are linear orders $<_1$ and $<_2$ that include $<$

and have $x <_1 y$ and $y <_2 x$. Following Dushnik and Miller

[11], we shall define the <u>dimension</u> $D(<)$ of a partial order

as the smallest cardinal number m such that $<$ equals the

intersection of m linear orders.*

It turns out that when X is countable, this notion

of dimension is essentially the same as that defined above,

namely that for $n > 1$, $D(<) \leq n$ if and only if there are

functions f_1, f_2, ..., f_n on X such that for all $x, y \in X$,

(5) $x < y$ if and only if $(\forall i)[f_i(x) < f_i(y)]$.

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*Ore [30] uses the term "order dimension" for the
Dushnik–Miller notion, and the term "product dimension"
for the following equivalent notion due to Hiraguchi [18].
The product dimension of a partial order $<$ is the least
cardinal m such that $<$ can be embedded, as a partial order,
in the cardinal product of m chains. It should be men-
tioned that these two equivalent notions of dimension of
a partial order bear no relation to the "height" or "length"
of a partial order, which is also called dimension in some
geometrical contexts.

This is proved in Baker-Fishburn-Roberts [6].

Figure 1 shows three partial orders for $|X| = 6$. In these Hasse diagrams, $x < y$ if and only if x lies below y and there is a connected path from x up to y, each of whose links goes upward. It is easy to see that $D(<_3) = 2$. For $<_3$ is not a linear order, and two linear orders whose intersection is $<_3$ are $b < c < e < d < f < a$ and $e < f < b < a < c < d$. It turns out that $D(<_1) = D(<_2) = 3$, and we shall return to these two partial orders below.

A number of characterizations of partial orders with $D \leq 2$ have been obtained by Dushnik and Miller [11] and Baker, Fishburn, and Roberts [5, 6]. We shall state several of these here, and a complete summary may be found in [6].

Perhaps the basic characterization is the following, due to Dushnik and Miller. Suppose $\sim$ is the incomparability relation corresponding to the partial order $(X, <)$. Then $D(<) \leq 2$ if and only if there is a conjugate partial order $<^*$ on X satisfying the following condition for all $x, y \in X$:

$$x \sim y \quad \text{if and only if} \quad x <^* y \text{ or } y <^* x.$$

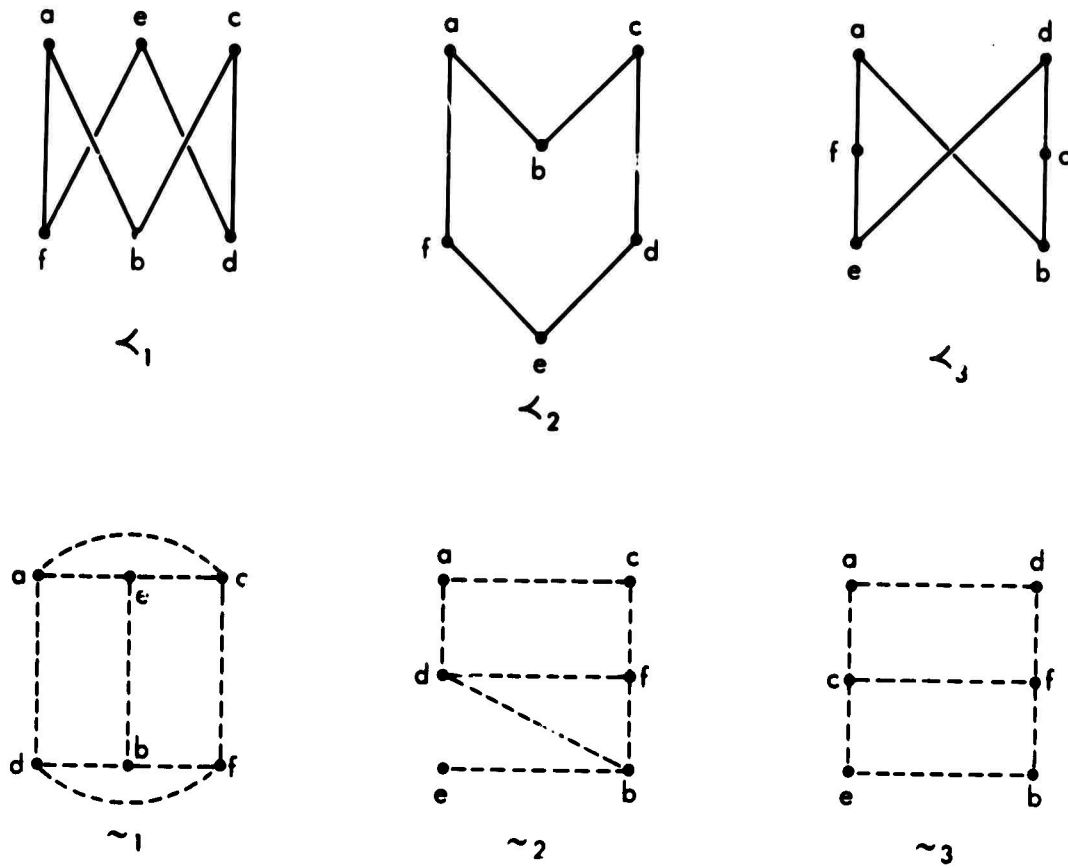Thus, for example, if X is the real plane and $<$ is defined on X by

Figure 1

$$(x_1, y_1) \prec (x_2, y_2)$$

if and only if

$$[x_1 < x_2 \text{ and } y_1 < y_2],$$

then $(X, \prec)$ has $D \leq 2$ because it has a conjugate partial order defined by

$$(x_1, y_1) \prec^* (x_2, y_2)$$

if and only if

$$[x_1 \geq x_2 \text{ and } y_1 \geq y_2 \text{ and } (x_1 \neq x_2 \text{ or } y_1 \neq y_2)].$$

To state a second characterization of the two–dimensional partial orders, we need the notion of a $\sim$–cycle. We say that $x_0, x_1, \ldots, x_{n-1}$ is a $\underline{\sim\text{–cycle}}$ if and only if $x_0 \sim x_1 \sim x_2 \sim \cdots \sim x_{n-1} \sim x_0$, and $x_i = x_j$ for $i \neq j$ implies $x_{i+1} \neq x_{j+1}$, where addition is taken modulo n.* A $\underline{\text{triangular chord}}$

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*It should be noted that this definition of cycle is slightly more general than the one usually given.

of such a cycle is a pair (x_i, x_{i+2}), where $x_i \sim x_{i+2}$ and where addition is again taken modulo n.

It is proved in Baker, Fishburn, and Roberts [5] that if $(X, <)$ is a partial order and $\sim$ is its incomparability relation, then $D(<) \leq 2$ if and only if every odd $\sim$-cycle has at least one triangular chord. We can now verify that $D(<_2) > 2$ for $<_2$ of Fig. 1. This follows because the $\sim_2$-cycle a, c, f, b, e, b, d has no triangular chord since $a \not\sim_2 f$, $c \not\sim_2 b$, $f \not\sim_2 e$, $b = b$, $e \not\sim_2 d$, $b \not\sim_2 a$, and $d \not\sim_2 c$. For a proof that $D(<_2)$ actually is 3, the reader is referred to Baker, Fishburn, and Roberts [6].

To get an even better picture of those partial orders which have dimension greater than two, let us define a <u>comparability cycle</u> in a partial order $(X, <)$ to be a sequence $x_0, x_1, \ldots, x_{2k} \in X$ of odd length $n = 2k + 1$ satisfying these three conditions for all i:

(i) x_i and x_{i+1} are comparable, i.e., $x_i < x_{i+1}$ or $x_{i+1} < x_i$ or $x_i = x_{i+1}$;

(ii) x_i and x_{i+k} are incomparable;

(iii)   if $x_i = x_j$ for some $j \neq i$, then $x_{i+k} \neq x_{j+k}$.

Here subscripts are interpreted modulo n. For each i,

x_{i+k} is one of the two elements which are "opposite" x_i in the cycle. It follows rather easily from the result about ∼cycles that a partial order $(X, <)$ has dimension at most two if and only if $(X, <)$ has no comparability cycle. (For a proof, see Baker, Fishburn, and Roberts [6].)

As an application, consider again the partial orders $<_1$, $<_2$ of Fig. 1. In both cases, the elements in alphabetical order make a comparability cycle if one element is repeated; e.g., a, a, b, c, d, e, f. Therefore both $<_1$ and $<_2$ have dimension greater than 2. (It is shown in Baker, Fishburn and Roberts [6] that in fact, $D(<_1) = D(<_2) = 3$.)

REFERENCES

1. Armstrong, W. E., "A Note on the Theory of Consumer's Behavior," Oxford Econ. Papers, 2 (1950), 119-122.

2. Armstrong, W. E., "The Determinateness of the Utility Function," Econ. J., 49 (1939), 453-467.

3. Armstrong, W. E., "Uncertainty and the Utility Function," Econ. J., 58 (1948), 1-10.

4. Armstrong, W. E., "Utility and the Theory of Welfare," Oxford Econ. Papers, 3 (1951), 259-271.

5. Baker, K. A., P. C. Fishburn, and F. S. Roberts, A New Characterization of Partial Orders of Dimension Two, The Rand Corporation, P-4339, March 1970; to appear in Proc. N.Y. Acad. Sci.

6. Baker, K. A., P. C. Fishburn, and F. S. Roberts, Partial Orders of Dimension 2, Interval Orders, and Interval Graphs, The Rand Corporation, P-4376, June 1970.

7. Birkhoff, G., Lattice Theory, 3d ed., American Mathematical Society, Providence, Rhode Island, 1967.

8. Block, H. D., and J. Marschak, "Random Orderings and Stochastic Theories of Responses," in I. Olkin, S. Ghurye, W. Hoeffding, W. Madow, and H. Mann (eds.), Contributions to Probability and Statistics, Stanford University Press, Stanford, California, 1959, pp. 97-132.

9. Dalkey, N. C., The Delphi Method: An Experimental Study of Group Opinion, The Rand Corporation, RM-5888-PR, June 1969.

10. Domotor, Z., Probabilistic Relational Structures and Their Applications, Institute for Mathematical Studies in the Social Sciences, Stanford University, Technical Report 144, 1969.

11. Dushnik, B., and E. W. Miller, "Partially Ordered Sets," Amer. J. Math., 63 (1941), 600-610.

12. Fishburn, P. C., Interval Orders, Mimeographed, 1969.

13. Fishburn, P. C., Intransitive Indifference in Preference Theory: A Survey, Operations Research, 18 (1970), 207-228.

14. Fishburn, P. C., "Intransitive Indifference with Unequal Indifference Intervals," J. Mathematical Psychology, 7 (1970), 144-149.

15. Fishburn, P. C., "Utility Theory," <u>Management Sci.</u>, <u>14</u> (1968), 335-378.

16. Fishburn, P. C., <u>Utility Theory for Decision Making</u>, Wiley & Sons, Inc., New York, 1970.

17. Friedman, M., "The Use of Ranks to Avoid the Assumption of Normality Implicit in the Analysis of Variance," <u>J. Amer. Statist. Assoc.</u>, <u>32</u> (1937), 675-699.

18. Hiraguchi, T., "On the Dimension of Orders," <u>Sci. Rep. Kanazawa Univ.</u>, <u>4</u> (1955), 1-20.

19. Holland, P. W., <u>Semiorder Theory I: Characterization of Semiorders</u>, Dept. of Statistics, Harvard University, NS 44, 1966.

20. Holland, P. W., <u>Semiorder Theory II: Linear Semiorders</u>, Dept. of Statistics, Harvard University, NS 45, 1966.

21. Kemeny, J. G., and J. L. Snell, <u>Mathematical Models in the Social Sciences</u>, Chap. II, "Preference Rankings: An Axiomatic Approach," Blaisdell Publishing Company, Waltham, Mass., 1962.

22. Kendall, M. G., <u>Rank Correlation Methods</u>, Hafner, New York, 1962.

23. Krantz, D. H., "Extensive Measurement in Semiorders," <u>Philos. Sci.</u>, <u>34</u> (1967), 348-362.

24. Krantz, D. H., "A Survey of Measurement Theory," in G. B. Dantzig and A. F. Veinott (eds.), <u>Mathematics of Decision Sciences</u>, Part II, Lectures in Applied Mathematics, Vol. 12, American Mathematical Society, Providence, Rhode Island, 1968.

25. Luce, R. D., "Semiorders and a Theory of Utility Discrimination," <u>Econometrica</u>, <u>24</u> (1956), 178-191.

26. Luce, R. D., and P. Suppes, "Preference, Utility, and Subjective Probability," in R. D. Luce, R. R. Bush, and E. Galanter (eds.), <u>Handbook of Mathematical Psychology, III</u>, John Wiley & Sons, Inc., New York, 1965, 249-410.

27. Marschak, J., "Actual versus Consistent Decision Behavior," <u>Behavioral Sci.</u>, <u>9</u> (1964), 103-110.

28. Milgram, A. N., "Partially Ordered Sets, Separating Systems and Inductiveness," in K. Menger (ed.), <u>Reports of a Mathematical Colloquium</u>, Second Series, No. 1, University of Notre Dame Press, South Bend, Indiana, 1939.

29. Miller, J. R., _Assessing Alternative Transportation Systems_, The Rand Corporation, RM-5865-DOT, April 1969.

30. Ore, O., _Theory of Graphs_, American Mathematical Society Colloquium Publications, Vol. 38 (1962).

31. Raiffa, H., _Preferences for Multi-attributed Alternatives_, The Rand Corporation, RM-5868-DOT/RC, April 1969.

32. Roberts, F. S., _Homogeneous Families of Semiorders and the Theory of Probabilistic Consistency_, The Rand Corporation, RM-5993-PR, October 1969; to appear in _J. Mathematical Psychology_.

33. Roberts, F. S., "Indifference Graphs," in F. Harary (ed.), _Proof Techniques in Graph Theory_, Academic Press, Inc., New York, 1969, 139-146.

34. Roberts, F. S.,"On Nontransitive Indifference," _J. Mathematical Psychology_, 7 (1970), 243-258.

35. Roberts, F. S., "On the Boxicity and the Cubicity of a Graph," in W. T. Tutte (ed.), _Recent Progress in Combinatorics_, Academic Press, Inc., New York, 1969.

36. Roberts, F. S., _On the Compatibility Between a Graph and a Simple Order_, The Rand Corporation, RM-5778-PR, September 1968; to appear in _J. Combinatorial Theory_.

37. Scott, D., and P. Suppes, "Foundational Aspects of Theories of Measurement," _J. Symbolic Logic_, 23 (1958), 113-128.

38. Suppes, P., and J. Zinnes, "Basic Measurement Theory," in R. D. Luce, R. R. Bush, E. Galanter (eds.), _Handbook of Mathematical Psychology_, I, John Wiley & Sons, Inc., New York, 1963, 1-76.

39. Szpilrajn, E., "Sur l'extension de l'ordre partiel," _Fund. Math._, 16 (1930), 386-389.

40. von Neumann, J., and O. Morgenstern, _Theory of Games and Economic Behavior_, Princeton University Press, Princeton, New Jersey 1953.